Application of Quaternions to Projective Geometry.

By C. H. Chapman.

Introduction.

In the following article I employ Hamilton's complex numbers* of the type $xi + yj + zk$. The real or complex quantities x, y, z are interpreted as the trilinear coordinates of a point in the plane of the triangle of reference. The properties of i, j and k† are utilized, but their possible and usual geometrical interpretation as directed lines in space of three dimensions is, for the present, left aside. The number $\rho = xi + yj + zk$ is spoken of as the affix of the point of which x, y, z are the trilinear coordinates; or, dualistically, $\alpha = ai + bj + ck$ is the affix of the line of which a, b, c are the coordinates.

The point equation of a line is $S\alpha\rho = 0$; here α is the affix of the line and ρ varies; but if ρ is constant and α varies, we obtain the pencil of lines through the point ρ. In that case $S\alpha\rho = 0$ is the equation of the point ρ. The equation of a line through the points α and β is

$$S.\rho V\alpha\beta = 0 \text{ or } S.\rho\alpha\beta = 0;$$

but this may also be interpreted as the equation of the point of intersection of the lines whose affixes are α and β.

The homogeneous equation of the second degree in three variables takes the form

$$S\rho\phi\rho = 0,$$

ϕ being the self-conjugate linear and vector function whose matrix is

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}.$$

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* Hankel, Complexe Zahlensysteme, p. 141.

† Tait's Quaternions, p. 36.

Starting from this, I show that the tangential equation of the conics is

$$S\alpha\phi^{-1}\alpha = 0,$$

which, for a pair of lines, becomes simply

$$(S.\sigma\tau_1)^2 = 0,$$

τ_1 being the affix of their point of intersection.

The relation of pole and polar is expressed by ϕ; if α is the affix of the pole, then $\phi\alpha$ is the affix of the polar. Conversely, given the polar, we pass to the pole by operating with ϕ^{-1}.

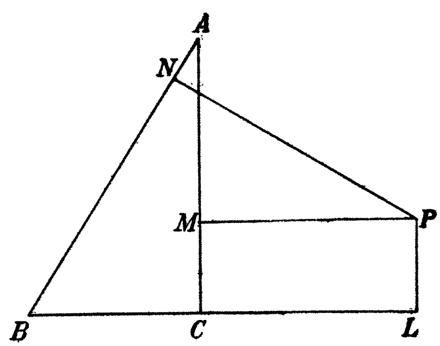
In case the matrix has one latent root equal to zero, the conic breaks up and ϕ^{-1} is indeterminate while ϕ is not. We can therefore pass from the pole to the polar, but not from polar to pole when the conic is a pair of lines.

In this new field, quite as much as in the more familiar ones, the calculus of quaternions shows its power by going directly to the result without analytical artifice, and by producing very simple formulas whose interpretation is intuitive. Especial attention might be called to the fact that the reciprocal polar relation is emphasized by the appearance of ϕ^{-1} in the tangential equation.

In Part 4 the method is extended to plane cubics, and there is nothing to hinder its application to curves in general, as I hope to show in a future paper.

Constant references are made to Tait's Quaternions for formulas, sometimes perhaps of a too elementary character to need reference. The second edition is used.

1.—*Equation of the Right Line.*



Consider a triangle ABC and a point P in its plane. From P let fall the perpendiculars PL, PM, PN on the sides of the triangle. There is no other point in the plane from which all the perpendiculars on the sides of the triangle will have the same lengths in the same order as those from P. Three perpendiculars chosen and arranged arbitrarily will not in general attach themselves in this way to any point, for the ends of any pair of them may be brought together, but the third one will then be found too short or too long. In fact the third is known when any two are given by virtue of the relation

$$p_1 a + p_2 b + p_3 c = 2K, \tag{1}$$

p_1, p_2, p_3 being the perpendiculars on the sides a, b, c, and K being the area of the triangle.

Each point in the plane has one set of perpendiculars attached to it, and no two points have the same set; but put

$$p_1 = lp_1', \; p_2 = mp_2', \; p_3 = np_3', \; n,$$

where l, m, n are any constants whatever, and make

$$al = A, \;\; bm = B, \;\; cn = C,$$

the relation (1) becomes

$$p_1' A + p_2' B + p_3' C = 2K,$$

and we conclude that there is a set of three numbers, proportional to arbitrary multiples of the perpendiculars on the sides of the triangle, attached to each point, and that no two points have the same set.

Let Hamilton's units, i, j, k, be attached as marks of distinction to distances measured along PL, PM, PN respectively; and let the perpendiculars from P, or fixed arbitrary multiples of them, be denoted by x, y, z; then the complex number $xi + yj + zk$ will be associated with the point P in such a way that when P is known the number is known, and conversely. The number $xi + yj + zk$ will be called the affix of the point P.

Numbers of this kind enjoy all the mathematical properties of vectors,* only the formulas obtained with them will now bear a new geometrical interpretation. They will be denoted, as is customary with vectors, by Greek letters

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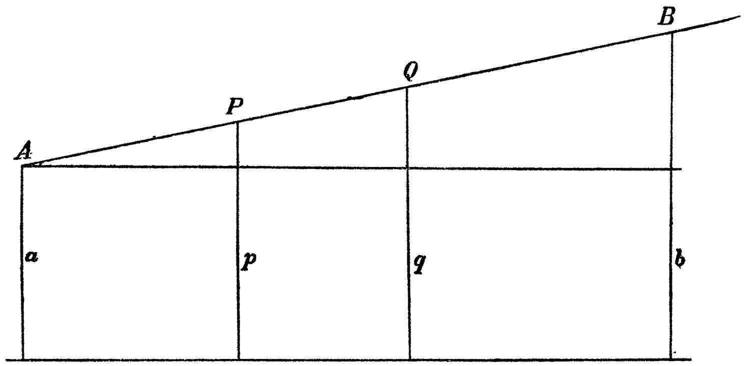
* Hankel, Complexe Zahlensysteme, §42.

If three points lie on a line, their affixes are connected by a linear homogeneous relation.



Let P and Q be the points lying on the line AB, and let the i-components of the affixes of A, P, Q be a, p, q respectively. We have from the figure $\frac{p-a}{q-a}=\frac{AP}{AQ}=\frac{\lambda}{\mu}$, say; whence $\mu p-\lambda q+(\lambda-\mu)a=0$. The same relation holds for the j and k components of the affixes of A, P, Q and hence for the affixes themselves. Conversely, if the affixes of three points are connected by a linear homogeneous relation, the points lie on a line.

Let $\alpha+m\beta+n\gamma=0$ be the relations between the affixes α, β, γ; we have only to make $\frac{\mu}{\lambda-\mu}=m$, $\frac{\lambda}{\mu-\lambda}=n$; thus a point is determined on the line through α and β whose affix is $\gamma=-\frac{\alpha+m\beta}{n}$.

Operating with $V.\beta\gamma$ and taking scalars, we deduce from the relation

$$\alpha+m\beta+n\gamma=0$$

the quaternion form

$$S.\alpha\beta\gamma=0, \tag{1}$$

which is therefore the necessary and sufficient condition that α, β, γ lie on a line.

Since the relation

$$(\alpha+\lambda\beta)-\alpha-\lambda\beta=0$$

is satisfied whatever λ may be, we conclude that the point $\alpha+\lambda\beta$ always lies on the line through α and β; and as λ varies through all real values, it represents successively all the points on the line.

The points α, β, γ lie on a line so long as (1) is satisfied; hence if

$$S.\rho V\alpha\beta=0=S.\rho\alpha\beta,$$

the locus of ρ is the straight line through α and β. Similarly, if $V\alpha\beta$ varies subject to (1), we shall have the equation of the straight line in the form

$$S.\alpha\rho = 0. \tag{2}$$

This line is the locus of all points γ given by the equation

$$\gamma = xV\sigma\alpha, \tag{3}$$

where x and σ may have any values, x being a scalar. We conclude from (2) that $V.\alpha\beta$ is the affix of the intersection of the lines $S.\alpha\rho = 0$, $S.\beta\rho = 0$.

The equations of the sides of the triangle of reference are

$$Si\rho = 0, \quad Sj\rho = 0, \quad Sk\rho = 0.$$

These equations are found by noting that the affix of the vertex B is pj; of C, qk; hence the equation of BC is $S.\rho Vjk = 0$; but $Vjk = jk = i$.

2.—*Transformation of Affixes.*

Let ρ be the affix of a point referred to a certain triangle; its affix referred to a triangle having δ_1, δ_2, δ_3 for the affixes of its vertices is required. We have

$$\rho S\delta_1\delta_2\delta_3 = \delta_1 S\rho\delta_2\delta_3 + \delta_2 S\rho\delta_1\delta_3 + \delta_3 S\rho\delta_2\delta_1.$$

Now $S.\rho\delta_2\delta_3$, ρ being any point, is proportional to the perpendicular from the point ρ on the line through δ_2 and δ_3. We may therefore write

$$\rho = X\delta_1 + Y\delta_2 + Z\delta_3,$$

where X, Y, Z are, for all points, proportional to the perpendiculars on the sides of the new triangle. Further than this, if $\delta_1 = ai + bj + ck$, then

$$\left[\frac{\delta_1}{\sqrt{a^2+b^2+c^2}}\right]^2 = -1;$$

so that, by including $\sqrt{a^2+b^2+c^2}$ and the corresponding factors for δ_2 and δ_3 in X, Y, Z, we may write

$$\rho = X\alpha + Y\beta + Z\gamma,$$

where $\alpha^2 = \beta^2 = \gamma^2 = -1$; although in general the system α, β, γ has not the properties of i, j, k. We shall have $\alpha\beta = -\beta\alpha$ if $S.\alpha\beta = 0$, and not otherwise; in case then $\alpha^2 = \beta^2 = \gamma^2 = -1$, and $\alpha\beta = -\beta\alpha$, $\alpha\gamma = -\gamma\alpha$, $\beta\gamma = -\gamma\beta$, we shall also have $\alpha\beta = \gamma$, $\beta\gamma = \alpha$, $\gamma\alpha = \beta$. The proof is just the same as it would be if α, β, γ represented a set of rectangular unit vectors. Such a system

of affixes may be called orthogonal. The affixes determined by the equation

$$(\phi - g)\rho = 0$$

form an orthogonal system if $\phi\rho$ is a self-conjugate* function linear in ρ.

We may also effect a linear transformation by writing $\rho = \psi\rho'$, where ψ is a matrix of order three.

3.—*Projective Geometry of Conics.*

The homogeneous equation of the conics

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0 \tag{1}$$

may be written in the form

$$S.\rho\phi\rho = 0, \tag{2}$$

where $\rho = xi + yj + zk$ and $\phi\rho = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}(xi + yj + zk)$.

Let τ and σ be the affixes of two points; any point on the line joining them has for its affix $\tau + \lambda\sigma$, λ being a scalar. Substituting this for ρ in (2) we shall obtain for λ the quadratic

$$S(\tau + \lambda\sigma)\phi(\tau + \lambda\sigma) = 0,$$

or

$$S.\tau\phi\tau + \lambda S.(\sigma\phi\tau + \tau\phi\sigma) + \lambda^2 S.\sigma\phi\sigma = 0.$$

But since ϕ is self-conjugate, this may be written

$$\lambda^2 S.\sigma\phi\sigma + 2\lambda S.\sigma\phi\tau + S.\tau\phi\tau = 0. \tag{3}$$

If the point τ be taken fixed and σ variable, then

$$S.\sigma\phi\tau = 0 \tag{4}$$

is the equation of a line, the polar line of τ with respect to the conic. This equation is symmetrical in σ and τ, as it should be.

Let $S.\sigma\alpha = 0$ be the equation of a polar line; then $\phi^{-1}\alpha$ is the affix of its pole. Now we know that

$$m\phi^{-1}\alpha = (m_1 - m_2\phi + \phi^2)\alpha;\dagger \tag{5}$$

hence $\phi^{-1}\alpha$ can always be determined unless $m = 0$; but m is the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

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*Tait, p. 90. †Tait, p. 82.

In case this determinant vanishes, the pole of the given line cannot be determined.

If the point τ lies on its own polar, it must satisfy equation (4), which becomes $S.\tau\phi\tau = 0$; hence τ also satisfies equation (2) and lies on the curve.

We had above $\tau = \phi^{-1}\alpha$; hence $S.\tau\phi\tau = S.\alpha\phi^{-1}\alpha = 0$. The condition

$$S.\alpha\phi^{-1}\alpha = 0 \tag{6}$$

being satisfied by the affixes of lines tangent to the conic, is the tangential equation of the conic.

We see that if τ is the affix of the pole, then $\phi\tau$ is the affix of the polar; and when $m = 0$, we can determine τ so that it shall satisfy the relation

$$\phi\tau_1 = 0. \tag{7}$$

This requires a nullity of order 1 in the matrix ϕ; in other words, one latent root of ϕ is zero.*

In this case it makes no difference what line in the plane is the locus of σ, we shall always have

$$S.\sigma\phi\tau_1 = 0;$$

so that any line may be taken for the polar of this exceptional point τ_1.

Let ω be any other point; its polar is given by

$$S.\omega\phi\rho = 0,$$

but this is satisfied when $\rho = \tau_1$, hence the polar lines of all points pass through τ_1.

We may write the equation of the polar of ω in the form

$$S.\rho\phi\omega = 0, \tag{8}$$

which is equally satisfied if we replace ω by $\omega + x\tau_1$. From this we conclude that any point on the line joining ω and τ_1 may be taken as the pole of the line (8).

The hypothesis still being that $m = 0$, let α be a point on the curve; any point on the line through τ_1 and α is $\tau_1 + x\alpha$. Let this be substituted in $S.\rho\phi\rho = 0$. We obtain

$$S.(\tau_1 + x\alpha)\,\phi\,(\tau_1 + x\alpha) = S.\tau_1\phi\tau_1 + 2xS.\alpha\phi\tau_1 + x^2S.\alpha\phi\alpha;$$

an expression in which every term vanishes. Hence the line joining τ_1 to any point on the curve forms part of the curve.

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* Taber, *Am. Journ. Math.*, Vol. XII, p. 362.

To determine the vector τ_1 which satisfies equation (7), we recall that $m = -S.\phi i\phi j\phi k$, if i, j, k be taken for a fundamental system.* Hence in this case

$$S.\phi i\phi j\phi k = S.i\phi V.\phi j\phi k = 0.$$

We have in all cases

$$S.j\phi V.\phi j\phi k = 0,$$
$$S.k\phi V.\phi j\phi k = 0.$$

Hence for any vector whatever ρ, we have

$$S.\rho\phi V.\phi j\phi k = 0,$$

from which we must conclude that

$$\phi V.\phi j\phi k = 0,$$

and finally that

$$\tau_1 = xV.\phi j\phi k. \tag{10}$$

We might show just as well that $\tau = xV.\phi k\phi i$, or $xV.\phi i\phi j$; hence, since the scalar equations for τ_1 are linear in the variables and it can therefore have only one value, we conclude that $V.\phi j\phi k = t_1 V.\phi k\phi i = t_2 V.\phi i\phi j$.

Let us now expand $V.\phi j\phi k$ in terms of i, j, k; assume

$$xV.\phi j\phi k = i + yj + zk,$$

a relation which must hold for some system of values of x, y, z. Operating with ϕ, we obtain the relation

$$\phi i + y\phi j + z\phi k = 0,$$

whence

$$yV.\phi j\phi k = -V.\phi i\phi k, \text{ or } y = \frac{1}{t_1}$$

and

$$zV.\phi k\phi j = -V.\phi i\phi j, \text{ or } z = \frac{1}{t_2}.$$

Therefore

$$xV.\phi j\phi k = i + \frac{1}{t_1}j + \frac{1}{t_2}k = x\tau_1. \tag{11}$$

In general, we may write for any vector α,

$$S.\phi i\phi j\phi k.\alpha = \phi iS.\alpha\phi j\phi k + \phi jS.\alpha\phi k\phi i + \phi kS.\alpha\phi i\phi j.$$

But

$$S.\phi i\phi j\phi k = -m,$$

hence, operating with ϕ^{-1},

$$-m\phi^{-1}\alpha = iS.\alpha\phi j\phi k + jS.\alpha\phi k\phi i + kS.\alpha\phi i\phi j.$$

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*Tait, p. 81.

From this we find

$$-mS.\alpha\phi^{-1}\alpha = S.\alpha iS.\alpha\phi j\phi k + S.\alpha jS.\alpha\phi k\phi i + S.\alpha kS.\alpha\phi i\phi j.$$

Now if $m = 0$, we may replace $V\phi j\phi k$ by τ_1 and we obtain the relation

$$S.\alpha\tau_1\left(S.\alpha i + \frac{1}{t_1}S.\alpha j + \frac{1}{t_2}S.\alpha k\right) = 0,$$

or

$$S.\alpha\tau_1 S.\alpha\left(i + \frac{1}{t_1}j + \frac{1}{t_2}k\right) = 0,$$

or finally $(S.\alpha\tau_1)^2 = 0$, by aid of (11).

This is the form assumed by the tangential equation when the conic breaks up into a pair of lines. It is the equation of the point τ_1 squared.

No line can be tangent to this conic unless it passes through τ_1; now the affix of any line may be written in the form

$$a\delta_1 + b\delta_2 + c\tau_1, \tag{12}$$

where δ_1 and δ_2 are the other two affixes determined by the equation $(\phi - g)\rho = 0$. These affixes form an orthogonal system.* Substituting (12) in the equation of the point τ_1, we obtain

$$S.\tau_1(a\delta_1 + b\delta_2 + c\tau_1) = 0,$$

or

$$c\tau_1^2 = 0,$$

whence, unless $\tau_1^2 = 0$, we conclude that $c = 0$. Any line therefore will be tangent to the pair of lines if its affix is a linear function of δ_1 and δ_2 only.

The curve

$$S.\rho\phi\rho - pS^2.\rho\phi\alpha = 0$$

is a conic touching the conic $S.\rho\phi\rho = 0$. It passes through the point α, if $p = \frac{1}{S.\alpha\phi\alpha}$, and its equation then becomes

$$S.\alpha\phi\alpha S\rho\phi\rho - S^2.\rho\phi\alpha = 0. \tag{13}$$

The points of contact are on the line $S.\rho\phi\alpha = 0$, which is the polar of α. Now we may write equation (13) in the form $S.\rho\,[\phi\rho S.\alpha\phi\alpha - \phi\alpha S.\rho\phi\alpha] = 0$; where the expression in the brackets is on its face a self-conjugate linear function of ρ. To compute for it the quantity m, we may choose α, $\phi\alpha$ and $V.\alpha\phi\alpha$ for a system

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* Tait, p. 89.

of independent affixes; then designating the function by ψ, we shall have

$$m = \frac{S.\psi\alpha\psi\phi\alpha\psi V.\alpha\phi\alpha}{S.\alpha\phi\alpha\, V.\alpha\phi\alpha};$$

but $\psi\alpha = \phi\alpha S.\alpha\phi\alpha - \phi\alpha S.\alpha\phi\alpha = 0$; hence for ψ the quantity m vanishes. We conclude that the conic represented by (13) is a pair of lines, the tangents from α to the conic.

The polar lines of the points whose affixes are α, β, γ are $S.\sigma\phi\alpha = 0$, $S.\sigma\phi\beta = 0$, $S.\sigma\phi\gamma = 0$; and if each point lies on the polars of the other two, we have in addition

$$S.\beta\phi\alpha = S.\beta\phi\gamma = 0;\ S.\alpha\phi\beta = S.\alpha\phi\gamma = 0;\ S.\gamma\phi\alpha = S.\gamma\phi\beta = 0. \tag{14}$$

The points α, β, γ are now the vertices of a polar triangle, and if we write $\rho = \frac{x\alpha}{(S.\alpha\phi\alpha)^{\frac{1}{2}}} + \frac{y\beta}{(S.\beta\phi\beta)^{\frac{1}{2}}} + \frac{z\gamma}{(S.\gamma\phi\gamma)^{\frac{1}{2}}}$, then the equation of the conic $S.\rho\phi\rho = 0$ takes the form

$$S.\left(\frac{x\alpha}{(S.\alpha\phi\alpha)^{\frac{1}{2}}} + \frac{y\beta}{(S.\beta\phi\beta)^{\frac{1}{2}}} + \frac{z\gamma}{(S.\gamma\phi\gamma)^{\frac{1}{2}}}\right)\phi\left(\frac{x\alpha}{(S.\alpha\phi\alpha)^{\frac{1}{2}}} + \frac{y\beta}{(S.\beta\phi\beta)^{\frac{1}{2}}} + \frac{z\gamma}{(S.\gamma\phi\gamma)^{\frac{1}{2}}}\right) = 0,$$

or

$$x^2 + y^2 + z^2 = 0. \tag{15}$$

Let $S.\rho\phi\rho = 0$ and $S.\rho\psi\rho = 0$ represent two conics. The polar of a point α referred to each respectively will be given by $S.\sigma\phi\alpha = 0$, $S.\sigma\psi\alpha = 0$. These equations will represent the same line if, and only if,

$$\phi\alpha = x\psi\alpha;\ \text{whence}\ \alpha = x\phi^{-1}\psi\alpha,$$

or $\alpha = \frac{1}{x}\psi^{-1}\phi\alpha$, since if α satisfies one of these equations it must also satisfy the other. There are three points, then, which have the same polar lines with respect to the two conics. Their affixes are given as the roots of

$$(\phi^{-1}\psi - x)\,\alpha = 0,$$

and they form a polar triangle common to the two conics. For let $\delta_1, \delta_2, \delta_3$ be the roots of $(\phi^{-1}\psi - x)\,\alpha = 0$, and h_1, h_2, h_3, supposed all different, the corresponding scalars; so that $\phi^{-1}\psi\delta_1 = h_1\delta_1$, $\phi^{-1}\psi\delta_2 = h_2\delta_2$; noting that $\psi\delta_1 = h_1\phi\delta_1$, we have now $S.\phi^{-1}\psi\delta_2\psi\delta_1 = h_1h_2S.\delta_2\phi\delta_1$. But $S.\phi^{-1}\psi\delta_2\psi\delta_1 = h_2S.\delta_2\psi\delta_1 = h_2S.\delta_1\psi\delta_2$ $= h_2^2S.\delta_1\phi\delta_2$, since $\psi\delta_2 = h_2\phi\delta_2$. We have then $h_1h_2S.\delta_2\phi\delta_1 = h_2^2S.\delta_2\phi\delta_1$, whence we conclude that since h_1 and h_2 are different, $S.\delta_2\phi\delta_1$ must vanish. Hence δ_2 is on

the polar of δ_1. The same is true of δ_3. In this way the triangle is seen to be a polar triangle. If the equation $S.\rho\phi\rho = 0$ referred to this triangle becomes

$$x^2 + y^2 + z^2 = 0,$$

then $S.\rho\psi\rho = 0$ becomes

$$x^2 S.\alpha\psi\alpha + y^2 S.\beta\psi\beta + z^2 S.\gamma\psi\gamma = 0. \qquad (16)$$

Let α and β be the affixes of any two lines; a line through their intersection has the affix $\alpha + \lambda\beta$. Substituting this in the tangential equation of the conic $S\alpha\phi^{-1}\alpha = 0$, we obtain

$$S.(\alpha + \lambda\beta)\,\phi^{-1}(\alpha + \lambda\beta) = 0;$$

that is, $$S.\alpha\phi^{-1}\alpha + 2\lambda S.\beta\phi^{-1}\alpha + \lambda^2 S.\beta\phi^{-1}\beta = 0. \qquad (17)$$

If λ_1, λ_2 are the roots of this equation, then

$$\alpha + \lambda_1\beta,\ \alpha + \lambda_2\beta \qquad (18)$$

are the affixes of the tangents to the conic through the intersection of the lines $S.\alpha\rho = 0$, $S.\beta\rho = 0$. The lines (18) form a harmonic pencil with α and β if

$$S.\beta\phi^{-1}\alpha = 0; \qquad (19)$$

that is, if β is the affix of lines passing through the pole of α. For we know that $\phi^{-1}\alpha$ is the affix of the pole when α is the affix of the polar. In fact (19) is the equation of the pole of α. If $S.\alpha\rho = 0$ passes through its own pole, we must have $S.\alpha\phi^{-1}\alpha = 0$, which means of course that α is tangent to the conic.

Let $S.\rho\phi\rho = 0$ and $S.\rho\psi\rho = 0$ be any two conics which do not break up. The conic $S.\rho(\phi - \lambda\psi)\rho = 0$, (20), passes through their points of intersection. For the function $\phi - \lambda\psi$, which is self-conjugate, the quantity m has the value $-S.[(\phi - \lambda\psi)\,i\,(\phi - \lambda\psi)\,j\,(\phi - \lambda\psi)\,k]$; hence, when m vanishes, λ satisfies a cubic. There are then three values of λ for which $S.\rho(\phi - \lambda\psi)\rho = 0$ breaks up into a pair of lines. The double points on these pairs are the vertices of a polar triangle common to all conics of the system (20). For, if

$$S.\rho(\phi - \lambda_1\psi)\rho = 0 \text{ and } S.\rho(\phi - \lambda_2\psi)\rho = 0$$

are any two conics of the system, the affix α of a vertex of their common polar triangle is given by the equation $(\phi - \lambda_1\psi)\,\alpha = x\,(\phi - \lambda_2\psi)\,\alpha$, which is possible in general only if $\phi\alpha = g\psi\alpha$; that is, the polar triangle common to any two conics of the system is the one common to $S.\rho\phi\rho = 0$ and $S.\rho\psi\rho = 0$. But the equation may also subsist if $(\phi - \lambda_1\psi)\,\alpha = 0$, $(\phi - \lambda_2\psi)\,\alpha = 0$; that is, if each of the two conics

breaks up into a pair of lines which intersect at α. Hence the three values of λ for which $S.\rho(\phi - \lambda\psi)\rho = 0$ breaks up, are the values of x for which $\phi\alpha = x\psi\alpha$; and the double points on the lines are the vertices of the polar triangle common to the whole system. Hence these double points have the affixes δ_1, δ_2, δ_3 which satisfy the equation

$$\phi\alpha = x\psi\alpha,$$

and are such that $\psi\delta_1 = h_1\phi\delta_1$, $\psi\delta_2 = h_2\phi\delta_2$, $\psi\delta_3 = h_3\phi\delta_3$, and the three values of λ are $\frac{1}{h_1}$, $\frac{1}{h_2}$, $\frac{1}{h_3}$ respectively.

When the conics are referred to δ_1, δ_2, δ_3 as a triangle of reference, their equations become

$$\left.\begin{aligned} x^2S.\delta_1\phi\delta_1 + y^2S.\delta_2\phi\delta_2 + z^2S.\delta_3\phi\delta_3 = 0,\\ x^2S.\delta_1\psi\delta_1 + y^2S.\delta_2\psi\delta_2 + z^2S.\delta_3\psi\delta_3 = 0.\end{aligned}\right\} \tag{21}$$

Remembering that $\psi\delta_1 = h_1\phi\delta_1$, etc., the second of equations (21) becomes

$$h_1x^2S.\delta_1\phi\delta_1 + h_2y^2S.\delta_2\phi\delta_2 + h_3z^2S.\delta_3\phi\delta_3 = 0.$$

Now, including $S.\delta_1\phi\delta_1$, $S.\delta_2\phi\delta_2$, $S.\delta_3\phi\delta_3$ in x^2, y^2, z^2 respectively, their final form is

$$\begin{aligned} x^2 + y^2 + z^2 &= 0,\\ h_1x^2 + h_2y^2 + h_3z^2 &= 0.\end{aligned}$$

These equations may be easily solved for the intersections of the two conics.

There is an interesting correspondence, which this treatment brings into prominence, between polar triangles in a conic, and systems of conjugate diameters in central surfaces of the second order. If the equation of two surfaces be $S.\rho\phi\rho = 1$ and $S.\rho\psi\rho = 1$, a system of diameters conjugate in both is given as the roots of $\phi^{-1}\psi\rho = x\rho$, which is also the equation that determines the vertices of the polar triangle common to the conics $S.\rho\phi\rho = 0$ and $S.\rho\psi\rho = 0$.*

4.—*Plane Cubics.*

If we write

$$\begin{aligned} -\epsilon &= a_1i + b_1j + c_1k\\ -\eta &= b_1i + b_2j + c_2k\\ -\gamma &= c_1i + c_2j + c_3k\\ -\beta &= b_2i + b_3j + c_4k\\ -\delta &= c_2i + c_4j + c_5k\\ -\varkappa &= c_3i + c_5j + c_6k\end{aligned}$$

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* Tait's Quaternions, Art. 269.

and
$$\rho = pi + qj + rk,$$
then the matrix
$$\begin{vmatrix} S.\epsilon\rho & S.\eta\rho & S.\gamma\rho \\ S.\eta\rho & S.\beta\rho & S.\delta\rho \\ S.\gamma\rho & S.\delta\rho & S.\varkappa\rho \end{vmatrix},$$
which may be denoted by ϕ_ρ, has the following properties:

1). $\phi_{\rho+\sigma} = \phi_\rho + \phi_\sigma$.
2). $S.\alpha\phi_\beta\beta = S.\beta\phi_\beta\alpha = S.\alpha\phi_\alpha\beta$.
3). $S.\gamma\phi_\alpha\beta = S.\beta\phi_\alpha\gamma = S.\alpha\phi_\beta\gamma = S.\alpha\phi_\gamma\beta$.
4). $\phi_\alpha\beta = \phi_\beta\alpha$.
5). $dS.\rho\phi_\rho\rho = 3S.d\rho\phi_\rho\rho$.

The equation
$$S.\rho\phi_\rho\rho = 0 \tag{1}$$
represents a plane cubic in general form; and if the content of ϕ_ρ be denoted by m_ρ, then
$$m_\rho = 0 \tag{2}$$
is the equation of the Hessian of the cubic. The curves 1) and 2) intersect in nine points. Let α be one of the nine; we have
$$S.\alpha\phi_\alpha\alpha = 0 \tag{3}$$
and
$$m_\alpha = 0. \tag{4}$$
There is consequently a point τ for which
$$\phi_\alpha\tau = 0. \tag{5}$$
If τ coincides with α, so that
$$\phi_\alpha\alpha = 0, \tag{6}$$
then α is a double point on the cubic.

To show this, we observe that the points where the line joining β and δ cuts the cubic are given by writing in $\beta + \lambda\delta$ the values of λ which satisfy
$$S.(\beta + \lambda\delta)(\phi_\beta + \lambda\phi_\delta)(\beta + \lambda\delta) = 0,$$
or
$$S.\beta\phi_\beta\beta + 3\lambda S.\delta\phi_\beta\beta + 3\lambda^2 S.\delta\phi_\beta\delta + \lambda^3 S.\delta\phi_\delta\delta = 0. \tag{7}$$
If $\beta = \alpha$ and $\phi_\alpha\alpha = 0$, two roots of this equation are equal to zero, whatever point δ may be. Hence α is a double point. All three roots vanish if δ lies on the conic $S.\delta\phi_\alpha\delta = 0$. But when $m_\alpha = 0$ this conic breaks up into a pair of lines. If then δ lies on either of these lines, the line joining α to δ meets the

cubic at α in three coincident points; but since $\phi_\alpha \alpha = 0$, α is the double point on the conic. Hence the tangents to the cubic at α are the lines into which the conic $S.\rho\phi_\alpha\rho$ breaks up.

If not only $\phi_\beta\beta = 0$, but also $\phi_\delta\delta = 0$, then equation (7) vanishes identically, and we conclude that the line joining β and δ forms part of the curve. If there are three double points given by the relations

$$\phi_\alpha\alpha = 0, \quad \phi_\beta\beta = 0, \quad \phi_\gamma\gamma = 0,$$

we may write $\rho = x\alpha + y\beta + z\gamma$, and this value substituted in

$$S.\rho\phi_\rho\rho = 0$$

reduces the equation to

$$xyz = 0. \tag{8}$$

Change δ to ρ in equation (7) and we see without difficulty that the polar line of β with respect to the cubic is given by

$$S.\rho\phi_\beta\beta = 0 \tag{9}$$

and the polar conic by

$$S.\rho\phi_\beta\rho = 0, \tag{10}$$

while (9) also gives the polar of β with respect to the conic (10). If β is on the cubic, (9) gives the tangent at β. The point ρ cannot traverse the loci (9) and (10) simultaneously unless the conic breaks up into two lines, one of which is the tangent to the cubic at β. But if, β being on the cubic, equations (9) and (10) are satisfied, β is a point of inflection. It is the polar conics of points of inflection therefore which break up into right lines; hence it is for points of inflection β that ϕ_β has the property indicated by the equation

$$\phi_\beta\tau = 0$$

for some point τ.

Let β_1, β_2 be two points of inflection, and τ_1, τ_2 two numbers such that

$$\phi_{\beta_1}\tau_1 = 0, \quad \phi_{\beta_2}\tau_2 = 0.$$

From the expression $V.\,V.\tau_1\beta_1 V.\tau_2\beta_2$ we derive the equation

$$\beta_1 S.\tau_1\tau_2\beta_2 - \tau_1 S.\beta_1\tau_2\beta_2 + \beta_2 S.\tau_2\tau_1\beta_1 - \tau_2 S.\beta_2\tau_1\beta_1 = 0, \tag{11}$$

which may be written

$$\beta_1 + y\tau_1 + z\beta_2 + w\tau_2 = 0, \tag{12}$$

where

$$y = -\frac{S.\beta_1\tau_2\beta_2}{S.\tau_1\tau_2\beta_2}; \quad z = -\frac{S.\tau_1\tau_2\beta_1}{S.\tau_1\tau_2\beta_2}; \quad w = -\frac{S.\tau_1\beta_1\beta_2}{S.\tau_1\tau_2\beta_2}.$$

By multiplying (12) by $\phi_{\beta_1}\beta_1$ and taking scalars, we find another value for z,

$$z = -\frac{S.\beta_2\phi_{\beta_1}\beta_1}{S.\beta_1\phi_{\beta_2}\beta_2}. \tag{13}$$

It is seen immediately that the line joining any two points β_1, β_2 on the cubic cuts the curve again in the point

$$\beta_3 = \beta_1 + z\beta_2, \tag{13'}$$

z having the value (13). Taking β_1 and β_2 points of inflection, let us find the effect of ϕ_{β_3} upon $\beta_1 + w\tau_2$. We have

$$\phi_{\beta_3}(\beta_1 + w\tau_2) = \phi_{\beta_1}\beta_1 + z\phi_{\beta_1}\beta_2 + w\phi_{\beta_1}\tau_2.$$

But from (12), $\qquad \phi_{\beta_1}\beta_1 + z\phi_{\beta_1}\beta_2 + w\phi_{\beta_1}\tau_2 = 0.$

Hence β_3 possesses the characteristic property of a point of inflection and is such that

$$\phi_{\beta_3}\tau_3 = 0,$$

where

$$\tau_3 = \beta_1 + w\tau_2 = -y\tau_1 - z\beta_2. \tag{14}$$

Upon the line joining β_1 and β_2 there can be no more than two points ρ such that

$$\phi_\rho\rho = p\rho, \tag{15}$$

where p is a scalar. For take $\rho = \beta_1 + t\beta_2$, then (15) becomes

$$\phi_{\beta_1}\beta_1 + 2t\phi_{\beta_1}\beta_2 + t^2\phi_{\beta_2}\beta_2 = p(\beta_1 + t\beta_2). \tag{16}$$

Multiplying both numbers by $V.\beta_1\beta_2$ and taking scalars, we find

$$S.\beta_1\beta_2\phi_{\beta_1}\beta_1 + 2tS.\beta_1\beta_2\phi_{\beta_1}\beta_2 + t^2S.\beta_1\beta_2\phi_{\beta_2}\beta_2 = 0, \tag{17}$$

a quadratic in t. There are no more than two values of t which satisfy (17).

If there are three points, ρ_1, ρ_2, ρ_3, mutually orthogonal and such that

$$\phi_{\rho_1}\rho_1 = p_1\rho_1;\quad \phi_{\rho_2}\rho_2 = p_2\rho_2;\quad \phi_{\rho_3}\rho_3 = p_3\rho_3, \tag{18}$$

then upon the line joining any two of them there lie three points of inflection.

It may be noted, to begin with, that $\phi_{\rho_1}\rho_2$ is orthogonal to both ρ_1 and ρ_2, since $S.\rho_1\phi_{\rho_1}\rho_2 = S.\rho_2\phi_{\rho_1}\rho_1 = p_1S.\rho_2\rho_1 = 0$; and

$$S.\rho_2\phi_{\rho_1}\rho_2 = S.\rho_1\phi_{\rho_2}\rho_2 = p_2S.\rho_1\rho_2 = 0.$$

It follows that

and in the same way,

$$\left.\begin{array}{l} \phi_{\rho_1}\rho_2 = f_3\rho_3; \\ \phi_{\rho_2}\rho_3 = f_1\rho_1;\quad \phi_{\rho_3}\rho_1 = f_2\rho_2. \end{array}\right\} \tag{19}$$

Moreover, since

$$S.\rho_3\phi_{\rho_1}\rho_2 = S.\rho_2\phi_{\rho_3}\rho_1 = S.\rho_1\phi_{\rho_3}\rho_2,$$

we must have

$$f_3\rho_3^2 = f_2\rho_2^2 = f_1\rho_1^2. \tag{20}$$

Let a number $\tau = x\rho_1 + y\rho_2 + z\rho_3$ be formed, and let us inquire if there is a number $\lambda\rho_1 + \mu\rho_2$ such that

$$\phi_{(\lambda\rho_1+\mu\rho_2)}\tau = 0;$$

that is, if

$$\lambda\phi_{\rho_1}\tau + \mu\phi_{\rho_2}\tau = 0. \tag{21}$$

Equation (21) gives the vector equation

$$\lambda(xp_1\rho_1 + yf_3\rho_3 + zf_2\rho_2) + \mu(xf_3\rho_3 + yp_2\rho_2 + zf_1\rho_1) = 0,$$

which yields the three scalar equations

$$\left.\begin{aligned} \lambda xp_1 + \mu zf_1 &= 0, \\ \lambda zf_2 + \mu yp_2 &= 0, \\ \lambda yf_3 + \mu xf_3 &= 0. \end{aligned}\right\} \tag{22}$$

These equations are consistent if $\frac{\mu}{\lambda}$ satisfies the cubic

$$\begin{vmatrix} \lambda p_1 & 0 & \mu f_1 \\ 0 & \mu p_2 & \lambda f_2 \\ \mu f_3 & \lambda f_3 & 0 \end{vmatrix} = 0, \tag{23}$$

or

$$-\lambda^3 p_1 f_2 f_3 = \mu^3 p_2 f_1 f_3,$$

whence

$$\frac{\mu}{\lambda} = -\sqrt[3]{\frac{p_1f_2}{p_2f_1}}; \quad -\omega\sqrt[3]{\frac{p_1f_2}{p_2f_1}}; \quad -\omega^2\sqrt[3]{\frac{p_1f_2}{p_2f_1}}, \tag{24}$$

where

$$\omega^3 = +1.$$

We may for brevity denote these values of $\frac{\mu}{\lambda}$ by

$$-c, \quad -\omega c, \quad -\omega^2 c,$$

where

$$c = \sqrt[3]{\frac{p_1f_2}{p_2f_1}}. \tag{25}$$

Equations (22) give now

$$\frac{x}{z} = -\frac{\mu}{\lambda}\,\frac{f_1}{p}; \quad \frac{y}{z} = -\frac{\lambda}{\mu}\,\frac{f_2}{p_2},$$

and we obtain for τ the three values

$$\left.\begin{aligned}\tau_1 &= +\ c\frac{f_1}{p_1}\rho_1 + \frac{1}{c}\ \frac{f_2}{p_2}\rho_2 + \rho_3,\\ \tau_2 &= +\ \omega c\frac{f_1}{p_1}\rho_1 + \frac{\omega^2}{c}\ \frac{f_2}{p_2}\rho_2 + \rho_3,\\ \tau_3 &= +\ \omega^2 c\frac{f_1}{p_1}\rho_1 + \frac{\omega}{c}\ \frac{f_2}{p_2}\rho_2 + \rho_3.\end{aligned}\right\} \tag{26}$$

The corresponding values of $\lambda\rho_1 + \mu\rho_2$ are proportional to

$$\beta_1 = \rho_1 - c\rho_2;\quad \beta_2 = \rho_1 - \omega c\rho_2;\quad \beta_3 = \rho_1 - \omega^2 c\rho_2. \tag{27}$$

If β_1, β_2, β_3 are on the cubic, they are points of inflection, since they have the characteristic property of points of inflection.

If β_1 is on the cubic, we must have

$$S.\beta_1\phi_{\beta_1}\beta_1 = 0.$$

This yields the equation

$$S.(\rho_1 - c\rho_2)(p_1\rho_1 - 2cf_3\rho_3 + c^2p_2\rho_2) = 0,$$

or

$$p_1\rho_1^2 - c^3p_2\rho_2^2 = 0. \tag{28}$$

Substituting the value of c from equation (25), this becomes

$$p_1\rho_1^2 - \frac{p_1f_2}{p_2f_1}p_2\rho_2^2 = 0.$$

Whence

$$f_1\rho_1^2 - f_2\rho_2^2 = 0,$$

which is known to be true by equation (20). In a similar way it may be shown that β_2 and β_3 are on the curve. Hence the theorem is proved.

Under this hypothesis we have

$$\frac{S.\beta_2\phi_{\beta_1}\beta_1}{S.\beta_1\phi_{\beta_2}\beta_2} = \frac{f_1\rho_1^2 - \omega f_2\rho_2^2}{f_1\rho_1^2 - \omega^2 f_2\rho_2^2} = \frac{1-\omega}{1-\omega^2} = -\omega, \text{ by aid of (20);}$$

and the value previously found for β_3, that is,

$$\beta_3 = \beta_1 - \frac{S.\beta_2\phi_{\beta_1}\beta_1}{S.\beta_1\phi_{\beta_2}\beta_2}\beta_2$$

becomes

$$\begin{aligned}\beta_3 = \beta_1 + \omega\beta_2 &= \rho_1 - c\rho_2 + \omega(\rho_1 - \omega c\rho_2) = (1+\omega)\rho_1 - c(1+\omega^2)\rho_2\\ &= -\omega^2\rho_1 + c\omega\rho_2 = -\omega^2\rho_1 + c\omega\rho_2 = -\frac{1}{\omega}(\rho_1 - c\omega^2\rho_2).\end{aligned}$$

This value of β_3 differs only by a constant factor from that found in equation (27); the results found so far are therefore consistent.

We had in equation (12)

$$w = -\frac{S.\tau_1\beta_1\beta_2}{S.\tau_1\tau_2\beta_2},$$

which, in terms of ρ_1, ρ_2, ρ_3, becomes

$$w = -\begin{vmatrix} c\dfrac{f_1}{p_1} & \dfrac{1}{c}\dfrac{f_2}{p_2} & 1 \\ 1 & -c & 0 \\ 1 & -\omega c & 0 \end{vmatrix} \div \begin{vmatrix} c\dfrac{f_1}{p_1} & \dfrac{1}{c}\dfrac{f_2}{p_2} & 1 \\ \omega c\dfrac{f_1}{p_1} & \dfrac{\omega^2}{c}\dfrac{f_2}{p_2} & 1 \\ 1 & -\omega c & 0 \end{vmatrix} = -\frac{c^2 p_2}{f_2(1+2\omega)}. \qquad (29)$$

With this value of w we have

$$\beta_1 + w\tau_2 = -\frac{f_2}{\omega(1-\omega)c^2p_2}\tau_3, \qquad (30)$$

which differs only by a constant factor from the value of τ_3 found in (26).

Arrived at this point we may now observe that taking

$$\rho_1 = x\beta_1 + y\beta_2,$$
$$\rho_2 = u\beta_1 + v\beta_2,$$

and c an undetermined scalar, it is possible to put β_1 and β_2 in the form

$$\left.\begin{aligned} \beta_1 &= \frac{v\rho_1 - y\rho_2}{xv - uy} = \rho_1 - c\rho_2, \\ \beta_2 &= \frac{x\rho_2 - u\rho_1}{xv - uy} = \rho_1 - \omega c\rho_2. \end{aligned}\right\} \qquad (31)$$

For, from the second and third members of equations (31), we find as a possible system of values,

$$\left.\begin{aligned} x &= -\frac{\omega}{1-\omega}, & y &= \frac{1}{1-\omega}, \\ u &= \frac{-1}{c(1-\omega)}, & v &= \frac{1}{c(1-\omega)}. \end{aligned}\right\} \qquad (32)$$

The arbitrary constant c may be so chosen that

$$\frac{S.\beta_2\phi_{\beta_1}\beta_1}{S.\beta_1\phi_{\beta_2}\beta_2} = -\omega, \qquad (33)$$

and then from equation (13′) we find

$$\beta_3 = \beta_1 + \omega\beta_2, \qquad (34)$$

which, by virtue of (31), becomes

$$\beta_3 = -\frac{1}{\omega}(\rho_1 - c\omega^2\rho_2), \tag{35}$$

which differs from the value of β_3 in (27) only by the factor $\omega - 1$.

The condition $$S.\rho_1\rho_2 = 0,$$
which takes the form

$$\beta_1^2 + \omega^2\beta_2^2 + \omega S.\beta_1\beta_2 = 0, \tag{36}$$

and does not contain c explicitly, may be satisfied if $\frac{T\beta_1}{T\beta_2}$ be properly chosen.

Let ρ_3 be a number satisfying the conditions

$$S.\rho_1\rho_3 = 0, \quad S.\rho_2\rho_3 = 0, \tag{37}$$

then ρ_1, ρ_2, ρ_3 form an orthogonal system.

We next notice that owing to the conditions

$$S.\tau_1\phi_{\beta_1}\beta_1 = 0, \quad S.\beta_1\phi_{\beta_1}\beta_1 = 0, \tag{38}$$

we may write

$$\phi_{\beta_1}\beta_1 = g_1 V.\tau_1\beta_1,$$

and similarly,

$$\left.\begin{aligned}\phi_{\beta_2}\beta_2 &= g_2 V.\tau_2\beta_2,\\ \phi_{\beta_3}\beta_3 &= g_3 V.\tau_3\beta_3.\end{aligned}\right\} \tag{39}$$

The conditions

$$\phi_{\beta_1}\tau_1 = \phi_{\beta_2}\tau_2 = \phi_{\beta_3}\tau_3 = 0 \tag{40}$$

yield three other vector equations.

Two more scalar conditions can be obtained from the fact that we must have

$$\beta_1 = \beta_2 - \frac{S.\beta_3\phi_{\beta_2}\beta_2}{S.\beta_2\phi_{\beta_3}\beta_3}\beta_3$$

and

$$\beta_2 = \beta_3 - \frac{S.\beta_1\phi_{\beta_3}\beta_3}{S.\beta_3\phi_{\beta_1}\beta_1}\beta_1, \tag{41}$$

whereas equation (34) gives

$$\left.\begin{aligned}\beta_1 &= \beta_3 - \omega\beta_2,\\ \beta_2 &= \frac{1}{\omega}(\beta_3 - \beta_1).\end{aligned}\right\} \tag{42}$$

The expressions for β_1 in (41) and (42) can differ only by a factor, and we conclude that

$$\beta_3 - \omega\beta_2 = x\left(\beta_2 - \frac{S.\beta_3\phi_{\beta_2}\beta_2}{S.\beta_2\phi_{\beta_3}\beta_3}\beta_3\right);$$

whence
$$x = -\omega\,; \quad \frac{S.\beta_3\phi_{\beta_2}\beta_2}{S.\beta_2\phi_{\beta_3}\beta_3} = \frac{1}{\omega} = \omega^2. \tag{43}$$

Similarly,
$$\frac{S.\beta_1\phi_{\beta_3}\beta_3}{S.\beta_3\phi_{\beta_1}\beta_1} = 1. \tag{44}$$

We shall now put τ_1, τ_2, τ_3 in the forms

$$\left.\begin{aligned} \tau_1 &= x_1\rho_1 + x_2\rho_2 + x_3\rho_3, \\ \tau_2 &= y_1\rho_1 + y_2\rho_2 + y_3\rho_3, \\ \tau_3 &= z_1\rho_1 + z_2\rho_2 + z_3\rho_3, \end{aligned}\right\} \tag{45}$$

and equations (39) become

$$\left.\begin{aligned} \phi_{\rho_1}\rho_1 + c^2\phi_{\rho_2}\rho_2 - 2c\phi_{\rho_1}\rho_2 &= [-(cx_1 + x_2)V.\rho_1\rho_2 + x_3V.\rho_3\rho_1 + cx_3V.\rho_2\rho_3]\,g_1, \\ \phi_{\rho_1}\rho_1 + \omega^2c^2\phi_{\rho_2}\rho_2 - 2\omega c\phi_{\rho_1}\rho_2 &= [-(\omega cy_1 + y_2)V.\rho_1\rho_2 + y_3V.\rho_3\rho_1 + \omega cy_3V.\rho_2\rho_3]\,g_2, \\ \phi_{\rho_1}\rho_1 + \omega c^2\phi_{\rho_2}\rho_2 - 2\omega^2c\phi_{\rho_1}\rho_2 &= [-(\omega^2cz_1 + z_2)V.\rho_1\rho_2 + z_3V.\rho_3\rho_1 + \omega^2cz_3V.\rho_2\rho_3]\,g_3. \end{aligned}\right\} \tag{46}$$

While from equations (40) we obtain

$$\left.\begin{aligned} x_1\phi_{\rho_1}\rho_1 - cx_2\phi_{\rho_2}\rho_2 + (x_2 - cx_1)\phi_{\rho_1}\rho_2 + x_3\phi_{\rho_1}\rho_3 - cx_3\rho_{\rho_2}\rho_3 &= 0, \\ y_1\phi_{\rho_1}\rho_1 - \omega cy_2\phi_{\rho_2}\rho_2 + (y_2 - \omega cy_1)\phi_{\rho_1}\rho_2 + y_3\phi_{\rho_1}\rho_3 - \omega cy_3\phi_{\rho_2}\rho_3 &= 0, \\ z_1\phi_{\rho_1}\rho_1 - \omega^2cz_2\phi_{\rho_2}\rho_2 + (z_2 - \omega^2cz_1)\phi_{\rho_1}\rho_2 + z_3\phi_{\rho_1}\rho_3 - \omega^2cz_3\phi_{\rho_2}\rho_3 &= 0. \end{aligned}\right\} \tag{47}$$

The fact that $\phi_{\beta_1}\beta_2$, $\phi_{\beta_2}\beta_3$, $\phi_{\beta_3}\beta_1$ are respectively multiples of $V.\tau_1\tau_2$, $V.\tau_2\tau_3$, $V.\tau_3\tau_1$ leads to the three following vector equations, in which h_1, h_2, h_3 are scalars:

$$\left.\begin{aligned} &\phi_{\rho_1}\rho_1 - (c + \omega c)\phi_{\rho_1}\rho_2 + \omega c^2\phi_{\rho_2}\rho_2 \\ &\quad = [(x_1y_2 - x_2y_1)V.\rho_1\rho_2 + (x_2y_3 - x_3y_2)V.\rho_2\rho_3 + (x_3y_1 - x_1y_3)V.\rho_3\rho_1]\,h_1, \\ &\phi_{\rho_1}\rho_1 - (\omega c + \omega^2c)\phi_{\rho_1}\rho_2 + c^2\phi_{\rho_3}\rho_2 \\ &\quad = [(y_1z_2 - y_2z_1)V.\rho_1\rho_2 + (y_2z_3 - y_3z_2)V.\rho_2\rho_3 + (y_3z_1 - y_1z_3)V.\rho_3\rho_1]\,h_2, \\ &\phi_{\rho_1}\rho_1 - (\omega^2c + c)\phi_{\rho_1}\rho_2 + \omega^2c^2\phi_{\rho_2}\rho_2 \\ &\quad = [(z_1x_2 - z_2x_1)V.\rho_1\rho_2 + (z_2x_3 - z_3x_2)V.\rho_2\rho_3 + (z_3x_1 - z_1x_3)V.\rho_3\rho_1]\,h_3. \end{aligned}\right\} \tag{48}$$

There will be no loss of generality by making

$$x_3 = y_3 = z_3 = 1 \tag{49}$$

in these equations. After doing this, multiply the first of equations (47) by 1, the second by ω and the third by ω^2 and add the results. By this process is obtained the vector equation

$$\left.\begin{aligned} &\phi_{\rho_1}\rho_1(x_1 + \omega y_1 + \omega^2z_1) - \phi_{\rho_2}\rho_2(x_2 + \omega^2y_2 + \omega z_2)\,c \\ &\qquad + \phi_{\rho_1}\rho_2\,[x_2 + \omega y_2 + \omega^2z_2 - c\,(x_1 + \omega^2y_1 + \omega z_1)] = 0, \end{aligned}\right\} \tag{50}$$

which yields the three scalar equations,

$$\left.\begin{aligned} &x_1 + \omega y_1 + \omega^2 z_1 = 0, \\ &x_2 + \omega^2 y_2 + \omega z_2 = 0, \\ &x_2 + \omega y_2 + \omega^2 z_2 - c(x_1 + \omega^2 y_1 + \omega z_1) = 0, \end{aligned}\right\} \tag{51}$$

unless in the exceptional case when

$$S.\phi_{\rho_1}\rho_1\ \phi_{\rho_2}\rho_2\ \phi_{\rho_1}\rho_2 = 0.$$

If we add the three equations (46), the sum of the left members is $3\phi_{\rho_1}\rho_1$, which is also the sum of the left numbers of (48); we may therefore equate the right members and thus obtain three more scalar equations:

$$\left.\begin{aligned} -g_1(cx_1 + x_2) - g_2(\omega c y_1 + y_2) - g_3(\omega^2 c z_1 + z_2) \\ = h_1(x_1 y_2 - x_2 y_1) + h_2(y_1 z_2 - y_2 z_1) + h_3(z_1 x_2 - z_2 x_1), \\ g_1 + g_2 + g_3 = h_1(x_2 - y_2) + h_2(y_2 - z_2) + h_3(z_2 - x_2), \\ cg_1 + \omega c g_2 + \omega^2 c g_3 = h_1(y_1 - x_1) h_2(z_1 - y_1) + h_3(x_1 - z_1). \end{aligned}\right\} \tag{52}$$

Again in (47) and (48) multiply the first, second and third equations in each group respectively by $1, \omega, \omega^2$; add and compare results. Three other scalar equations are obtained—

$$\left.\begin{aligned} -\omega g_1(cx_1 + x_2) - \omega^2 g_2(\omega c y_1 + y_2) - g_3(\omega^2 c z_1 + z_2) \\ = h_1(x_1 y_2 - x_2 y_1) + \omega h_2(y_1 z_2 - y_2 z_1) + \omega^2 h_3(z_1 x_2 - z_2 x_1), \\ \omega g_1 + \omega^2 g_2 + g_3 = h_1(x_2 - y_2) + \omega h_2(y_2 - z_2) + \omega^2 h_3(z_2 - x_2), \\ c\omega g_1 + cg_2 + c\omega^2 g_3 = h_1(y_1 - x_1) + \omega h_2(z_1 - y_1) + \omega^2 h_3(x_1 - z_1). \end{aligned}\right\} \tag{53}$$

Finally, using $\omega^2, \omega, 1$ as multipliers, adding and comparing, we derive three more scalar equations:

$$\left.\begin{aligned} \omega^2 g_1(cx_1 + x_2) + \omega g_2(\omega c y_1 + y_2) + g_3(\omega^2 c z_1 + z_2) \\ = 2h_1(x_1 y_2 - x_2 y_1) + 2\omega^2 h_2(y_1 z_2 - y_2 z_1) + 2\omega h_3(z_1 x_2 - z_2 x_1), \\ -\omega^2 g_1 - \omega g_2 - g_3 = 2h_1(x_2 - y_2) + 2\omega^2 h_2(y_2 - z_2) + 2\omega h_3(z_2 - x_2), \\ -\omega^2 c g_1 - \omega^2 c g_2 - \omega^2 c g_3 = 2h_1(y_1 - x_1) + 2\omega^2 h_2(z_1 - y_1) + 2\omega h_3(x_1 - z_1). \end{aligned}\right\} \tag{54}$$

Equations (51), (52), (53), and (54), twelve in number, enable us to determine eleven of the unknown quantities

$$\begin{array}{ccc} x_1, & y_1, & z_1, \\ x_2, & y_2, & z_2, \\ g_1, & g_2, & g_3, \\ h_1, & h_2, & h_3 \end{array}$$

in terms of the twelfth one.

To obtain the complete solution of these equations, which we know to be consistent, is apart from the present purpose. It will suffice to know a single system of values of the unknown quantities, and, taking a hint from equations (26), let us inquire if

$$\left.\begin{aligned} x_1 &= ca_1, & y_1 &= c\omega a_1, & z_1 &= c\omega^2 a_1, \\ x_2 &= \frac{a_2}{c}, & y_2 &= \frac{\omega^2 a_2}{c}, & z_2 &= \frac{\omega a_2}{c}, \end{aligned}\right\} \tag{55}$$

where a_1 and a_2 are to be determined, are possible forms for these quantities. Substituting them in equations (51), the first two are identically satisfied, while the third becomes

$$\frac{3a_2}{c} - 3c^2 a_1 = 0,$$

or

$$a_2 = c^3 a_1, \tag{56}$$

as we should have expected from equation (25). The last two equations in (52), (53) and (54) become respectively

$$\left.\begin{aligned} g_1 + g_2 + g_3 &= a_1 [h_1(\omega - 1) + h_2(\omega^2 - \omega) + h_3(1 - \omega^2)], \\ g_1 + \omega g_2 + \omega^2 g_3 &= \frac{a_2}{c} [h_1(1 - \omega^2) + h_2(\omega^2 - \omega) + h_3(\omega - 1)], \end{aligned}\right\} \tag{52'}$$

$$\left.\begin{aligned} \omega g_1 + \omega^2 g_2 + g_3 &= a_1 [h_1(\omega - 1) + h_2(1 - \omega^2) + h_3(\omega^2 - \omega)], \\ \omega g_1 + g_2 + \omega^2 g_3 &= \frac{a_2}{c} [h_1(1 - \omega^2) + h_2(1 - \omega^2) + h_3(1 - \omega^2)], \end{aligned}\right\} \tag{53'}$$

$$\left.\begin{aligned} -(\omega^2 g_1 + \omega g_2 + g_3) &= \frac{2a_2}{c} [h_1(1 - \omega^2) + h_2(\omega - 1) + h_3(\omega^2 - \omega)], \\ -\omega^2(g_1 + g_2 + g_3) &= 2a_1 [h_1(\omega - 1) + h_2(\omega - 1) + h_3(\omega - 1)]. \end{aligned}\right\} \tag{54'}$$

The values of $g_1, \ldots, h_3$ can now be found from equations (33), (43) and (44), together with

$$\left.\begin{aligned} \phi_{\beta_1}\beta_2 &= h_1 V.\tau_1\tau_2 \\ -\omega\phi_{\beta_2}\beta_3 &= h_2 V.\tau_2\tau_3 \\ -\omega\phi_{\beta_3}\beta_1 &= h_3 V.\tau_3\tau_1. \end{aligned}\right\} \tag{48'}$$

From equations (33) and (46) we find

$$-\omega = \frac{g_1(cx_3 - \omega cx_3)}{g_2(\omega cy_3 - cy_3)} = -\frac{g_1}{g_2}. \tag{58}$$

From (43) and (46), together with (35),

$$\omega^2 = \frac{-\frac{1}{\omega} g_2 (\omega c y_3 - \omega^2 c y_3)}{\frac{1}{\omega^2} g_3 (\omega^2 z_3 - \omega c z_3)} = + \omega \frac{g_2}{g_3}. \tag{59}$$

From (44) and (46),

$$1 = \frac{\frac{1}{\omega^2} g_3 (\omega^2 c z_3 - c z_3)}{-\frac{1}{\omega} g_1 (c x_3 - \omega^2 c x_3)} = + \omega^2 \frac{g_3}{g_1}. \tag{60}$$

Equations (58), (59), (60) are consistent without regard to c; we have therefore

$$\frac{g_2}{g_1} = \omega^2; \quad \frac{g_3}{g_1} = \omega. \tag{61}$$

From (48′) we find that

$$h_1 = \frac{S.\beta_2 \phi_{\beta_1} \beta_1}{S.\beta_1 \tau_1 \tau_2} = \frac{g_1 (c x_3 - \omega c x_3)}{x_2 y_3 - y_2 x_3 + c (x_1 y_3 - x_3 y_1)}$$

$$-\frac{1}{\omega} h_2 = \frac{S.\beta_3 \phi_{\beta_2} \beta_2}{S.\beta_2 \tau_2 \tau_3} = \frac{-\frac{1}{\omega} g_2 (\omega c y_3 - \omega^2 c y_3)}{y_2 z_3 - z_2 y_3 + \omega c (y_1 z_3 - y_3 z_1)}$$

$$-\frac{1}{\omega} h_3 = \frac{S.\beta_1 \phi_{\beta_3} \beta_3}{S.\beta_3 \tau_3 \tau_1} = \frac{\frac{1}{\omega^2} g_3 (\omega^2 c z_3 - c z_3)}{-\frac{1}{\omega} [z_2 x_3 - x_2 z_3 + \omega^2 c (z_1 x_3 - z_3 x_1)},$$

or more simply,

$$h_1 = \frac{g_1 c (1 - \omega)}{x_2 - y_2 + c (x_1 - y_1)},$$

$$h_2 = \frac{-g_2 c \omega (\omega - 1)}{y_2 - z_2 + \omega c (y_1 - z_1)},$$

$$h_3 = \frac{g_3 c (\omega + 1)(\omega - 1)}{z_2 - x_2 + \omega^2 c (z_1 - x_1)}.$$

By substituting here the values (55) assumed for $x_1, \ldots, z_2$, we have

$$\left. \begin{aligned} h_1 &= \frac{g_1 c^2 (1 - \omega)}{3 a_2}, \\ h_2 &= \frac{-g_2 c^2 (\omega - 1) \omega^2}{3 a_2}, \\ h_3 &= \frac{-g_3 c^2 (\omega - 1) \omega}{3 a_2}. \end{aligned} \right\} \tag{62}$$

We conclude that

$$\left.\begin{aligned} \frac{a_2}{c}\,\frac{h_1}{g_1} &= \frac{c}{3}(1-\omega), \\ \frac{a_2}{c}\,\frac{h_2}{g_1} &= -\frac{c}{3}(\omega-1)\,\omega = \frac{c}{3}\,\omega(1-\omega), \\ \frac{a_2}{c}\,\frac{h_3}{g_1} &= -\frac{c}{3}\,\omega^2(\omega-1) = \frac{c}{3}\,\omega^2(1-\omega). \end{aligned}\right\} \tag{63}$$

With these values, equations (52′), (53′), (54′) are all verified, provided that we impose upon c the condition

$$c = 1. \tag{64}$$

This leads to the conclusion that, equation (56),

$$a_1 = a_2, \tag{65}$$

while g_1 is undetermined. We have thus

$$\left.\begin{aligned} \frac{h_1}{g_1} &= \frac{1}{3a_2}(1-\omega), \\ \frac{h_2}{g_1} &= \frac{1}{3a_2}\,\omega(1-\omega), \\ \frac{h_3}{g_1} &= \frac{1}{3a_2}\,\omega^2(1-\omega). \end{aligned}\right\} \tag{66}$$

By these values of $g_2, \ldots\ldots, h_3, c$, the first equations of (52), (53) and (54) are reduced to identities without regard to the values of g_1 and a_2.

Let us now return to equations (46) and add them together, keeping in mind the values found for the scalar coefficients. We thus obtain

$$\phi_{\rho_1}\rho_1 = g_1 V.\rho_2\rho_3. \tag{67}$$

Again add them, after multiplying them in order by 1, ω, ω^2, and we have

$$\phi_{\rho_2}\rho_2 = g_1 V.\rho_3\rho_1. \tag{68}$$

Finally, multiply by ω^2, ω, 1 and add, the result is

$$\phi_{\rho_1}\rho_2 = a_2 g_1 V.\rho_1\rho_2. \tag{69}$$

Adding equations (47) and using (68), we find

$$\phi_{\rho_1}\rho_3 = a_2\phi_{\rho_2}\rho_2 = a_2 g_1 V.\rho_3\rho_1. \tag{70}$$

Multiply equations (47) in order by 1, ω^2, ω and add. This gives

$$\phi_{\rho_2}\rho_3 = a_2\phi_{\rho_1}\rho_1 = a_2 g_1 V.\rho_2\rho_3. \tag{71}$$

From (70) and (71) we learn that

$$S.\rho_1\phi_{\rho_3}\rho_3 = S.\rho_2\phi_{\rho_3}\rho_3 = 0\,,$$

and that consequently

$$\phi_{\rho_3}\rho_3 = x V.\rho_1\rho_2\,. \tag{72}$$

Since $V.\rho_1\rho_2$, $V.\rho_2\rho_3$, $V.\rho_3\rho_1$ are respectively proportional to ρ_3, ρ_1, ρ_2, the existence of three numbers having the property defined by equations (18) is demonstrated. The points ρ_1, ρ_2, ρ_3 form a triangle upon each side of which lie three points of inflection. Their affixes are given by the following scheme:

$$\left.\begin{aligned} \beta_1 &= \rho_1 - \rho_2\,, \\ \beta_2 &= \rho_1 - \omega\rho_2\,, \\ \beta_3 &= -\frac{1}{\omega}(\rho_1 - \omega^2\rho_2)\,, \end{aligned}\right\}$$

on the side joining ρ_1 and ρ_2.

$$\left.\begin{aligned} \gamma^1 &= \rho_2 - \rho_3\,, \\ \gamma^2 &= \rho_2 - \omega\rho_3\,, \\ \gamma_3 &= -\frac{1}{\omega}(\rho_2 - \omega^2\rho_3)\,, \end{aligned}\right\}$$

on the side joining ρ_2 and ρ_3.

$$\left.\begin{aligned} \delta_1 &= \rho_3 - \rho_1\,, \\ \delta_2 &= \rho_3 - \omega\rho_1\,, \\ \delta_3 &= -\frac{1}{\omega}(\rho_3 - \omega^2\rho_1)\,, \end{aligned}\right\}$$

on the side joining ρ_3 and ρ_1.

We find by solving these equations that

$$\left.\begin{aligned} \rho_1 &= \tfrac{1}{3}(\beta_1 + \beta_2 - \omega\beta_3)\,, \\ \rho_2 &= \tfrac{1}{3}(\gamma_1 + \gamma_2 - \omega\gamma_3)\,, \\ \rho_3 &= \tfrac{1}{3}(\delta_1 + \delta_2 - \omega\delta_3)\,. \end{aligned}\right\} \tag{73}$$

It is to be concluded that when the arrangement of the points of inflection on the sides of a triangle is known, the affixes of the vertices will be given by equations similar to (73), and those affixes will have the properties given in equations (18). It follows that the equation

$$\phi_\rho\rho = x\rho \tag{74}$$

has at least twelve solutions which fall into four sets with three members in a set.

In equation (72) the scalar x is undetermined, but its value is easily found. There is a number $\sigma_1 = l\rho_1 + m\rho_2 + n\rho_3$ which satisfies the equation

$$\phi_{\gamma_1}\sigma_1 = 0. \tag{75}$$

This gives us without difficulty, since $\gamma_1 = \rho_2 - \rho_3$,

$$\left.\begin{aligned} la_2g_1 - nx &= 0, \\ m - la_2 &= 0, \\ n - m\ \ &= 0, \end{aligned}\right\} \tag{76}$$

whence

$$x = g_1. \tag{77}$$

The equation of the cubic referred to the triangle ρ_1, ρ_2, ρ_3 takes the form

$$x^3 + y^3 + z^3 + 6a_2xyz = 0, \tag{78}$$

and we conclude that a_2 is the absolute invariant of the ternary cubic.*

From equations (67), (68), (72) we have

$$S.\rho_1\phi_{\rho_1}\rho_1 = S.\rho_2\phi_{\rho_2}\rho_2 = S.\rho_3\phi_{\rho_3}\rho_3 = g_1S.\rho_1\rho_2\rho_3. \tag{79}$$

Taking $\alpha = x\rho_1$, $\beta = y\rho_2$, $\gamma = z\rho_3$, we shall have

$$\phi_{\rho_1}\rho_1 = \frac{1}{x^2}\,\phi_\alpha\alpha = g_1V.\rho_2\rho_3 = \frac{g_1}{yz}\,V.\beta\gamma;$$

whence

$$\left.\begin{aligned} \phi_\alpha\alpha &= \frac{g_1x^2}{yz}\,V.\beta\gamma, \\ \phi_\beta\beta &= \frac{g_1y^2}{zx}\,V.\gamma\alpha, \\ \phi_\gamma\gamma &= \frac{g_1z^2}{xy}\,V.\alpha\beta. \end{aligned}\right\} \tag{80}$$

and similarly,

It will be seen that any one of the choices

$$\left.\begin{aligned} &x,\ y = \omega x\ ,\ z = \omega^2x, \\ &x,\ y = \omega^2x,\ z = \omega x\ , \\ &x,\ y = x\ \ ,\ z = x \end{aligned}\right\} \tag{81}$$

preserves the property of equations (79),

$$S.\alpha\phi_\alpha\alpha = S.\beta\phi_\beta\beta = S.\gamma\phi_\gamma\gamma. \tag{82}$$

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* Cf. Clebsch, Vorlesungen, p. 510.